The Dissertation Committee for Ray Yang
certifies that this is the approved version of the following dissertation:

Optimal regularity and nondegeneracy for minimizers of an energy related to the fractional Laplacian

Committee:

Luis Caffarelli, Supervisor

Ari Arapostathis

William Beckner

Richard Tsai

Alexis Vasseur

Optimal regularity and nondegeneracy for minimizers of an energy related to the fractional Laplacian

by

Ray Yang, A.B.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2011

Acknowledgments

I would like to thank my advisor Luis Caffarelli for his guidance and support during the years I spent in Austin.

The environment at the University of Texas at Austin has been a wonderful place for a young mathematician. I have learned much from my professors, who have been very patient and kind, from my fellow students, who have been enthusiastic and friendly, and from the staff, who have been helpful and cheering. Among my fellow students, I would like in particular to thank Lan Tang, Nestor Guillen, Hector Chang, Veronica Quitalo, Mark Lewko, and Paul Fili for many discussions, about mathematics and life alike.

Finally, I wish to thank my parents, who have always supported my interest in mathematics.

Optimal regularity and nondegeneracy for minimizers of an energy related to the fractional Laplacian

Publication No. __________________

Ray Yang, Ph.D.
The University of Texas at Austin, 2011

Supervisor: Luis Caffarelli

We study the optimal regularity and nondegeneracy of a free boundary problem related to the fractional Laplacian through the extension technique of Caffarelli and Silvestre [7]. Specifically, we show that minimizers of the energy

$$\int y^{1-2\sigma}|\nabla u|^2 dxdy + \int_{\{y=0\}} u^\gamma dx$$

where $x \in \mathbb{R}^n$ and $y \in [0, \infty)$ with $0 < \gamma < 1$, with free behavior on the set $\{y = 0\}$, are Holder continuous with exponent $\beta = \frac{2\sigma}{2-\gamma}$. These minimizers exhibit a free boundary: along $\{y = 0\}$, they divide into a zero set $\{u = 0\}$ and a positivity set where $\{u > 0\}$; we call the interface between these sets the free boundary.

The regularity is *optimal*, due to the non-degeneracy property of the minimizers: in any ball of radius r centered at the free boundary, the minimizer grows (in the supremum sense) like r^β.

This work is related to, but addresses a different problem from, recent work of Caffarelli, Roquejoffre, and Sire [5].

Table of Contents

Chapter 1

Introduction

This work addresses local properties (regularity and nondegeneracy) near the free boundary for minimizers of an energy functional related to the fractional Laplace operator with a penalty term. The work is inspired by recent work [5] on a similar problem, and we explain the broader background first.

In the theory of one-phase free boundary problems arising from the minimization of energy for the *classical* Laplacian, Alt and Caffarelli [2] analyzed minimizers of the energy $J(u) = \int (\nabla u)^2 + \chi_{u>0} dx$ subject to non-negative Dirichlet data, while the study of the free boundary arising from minimizers of the energy $J(u) = \int \frac{(\nabla u)^2}{2} + u dx$ with non-negative Dirichlet data is encompassed by the study of the obstacle problem. An intermediate case is the case studied by Alt and Phillips [3], which is that of the free boundary for minimizers of $J(u) = \int \frac{(\nabla u)^2}{2} + u^\gamma dx$, where $0 < \gamma < 1$. In a heuristic sense, we can view the Alt-Caffarelli problem as the case of $\gamma = 0$, and the case $\gamma = 1$ as a special case of the obstacle problem.

The problem we study is the analogue of the problem of Alt and Phillips for the fractional, rather than standard, Laplace operator. The current work only covers the regularity and nondegeneracy of energy minimizers, and is thus properly the analogue of Phillips' work in [19] and part of [18].

The fractional Laplace is a nonlocal integral operator, taking the form

$$(-\Delta)^{\sigma} u(x) = C_{n,\sigma} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2\sigma}} dz.$$

This operator has a corresponding energy given by

$$J(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(z))^2}{|x - z|^{n+2\sigma}} dz dx.$$

This latter term is a nonlocal energy, and not very easy to manipulate. In [7], Caffarelli and Silvestre introduced the notion of extension to one extra spatial dimension and examining a particular PDE on the upper half-space, with the fractional Laplacian being equivalent to the Dirichlet-to-Neumann map at the boundary. To be precise:

Theorem 1.0.1 (Caffarelli-Silvestre Extension Result)**.** *Consider* $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+$ *and a function u on the upper half-space satisfying*

$$\nabla \cdot (y^{1-2\sigma} \nabla u) = 0.$$

Then

$$(-\Delta)^{\sigma} u(x, 0) = -C_{n,\sigma} \lim_{y \to 0} y^{1-2\sigma} \partial_y u(x, y).$$

The natural energy corresponding to the second-order equation on the half-space is then $J(u) = \int y^{1-2\sigma} |\nabla u(x, y)|^2 dx dy$. This is an energy where it is easier to study the purely local properties of its minimizers.

The extension characterization of the fractional Laplacian has been used to study both the obstacle problem ([9]) and the equivalent $\gamma = 0$ case ([5]). We will apply it to study the intermediary case, that is to say, minimizers of energies

$$J(u) = \int y^{1-2\sigma}|\nabla u(x,y)|^2 dxdy + \int_{\{y=0\}} u^\gamma dx$$

in subsets of the upper half-space with parts of their boundary lying along $y = 0$, where we have $0 < \gamma < 1$. Since this is a study of the one-phase problem, we assume non-negative Dirichlet boundary conditions. The second term in the energy penalizes non-zero values of u along the hyperplane $\{y = 0\}$. Hence, we can consider separately the zero set of u (called the *contact set*), and its positivity set. Restricted to $\{y = 0\}$, the interface between the two is the *free boundary*.

We prove that in any neighborhood of a free boundary point, our energy minimizer u lies in the class C^β with a Holder seminorm that depends only on the distance to the free boundary, where $\beta = \frac{2\sigma}{2-\gamma}$ is the critical scaling exponent for the problem. This is called the *optimal regularity* of u, since we also prove the *non-degenerate* nature of u, namely, that in any ball of radius r about a free boundary point, $\sup_{B_r} u \geq Cr^\beta$ for a constant C that depends only on n, σ, and γ.

In the course of proving optimal regularity, we prove and use an improvement on the boundary Harnack inequality of Caffarelli, Fabes, Mortola, and Salsa [4], which may be of interest even to those not working in free boundaries.

The ideas behind the proof of the optimal regularity of energy minimizers for fractional-order cases can be extended to a proof of optimal regularity for the second-order case, which was first proved by Phillips [19]. Since the proof for the second-order case illustrates the ideas in a less involved setting than the fractional-order case, we provide it as well. The key ingredient for optimal regularity is the construction of a lower barrier, or subsolution, for the energy minimizer which is strictly positive at the center of a ball when the values near-by are "too large," thus, for a free boundary point to exist, the growth cannot be too great.

The organization of this thesis is as follows: in chapter 2, we provide a proof of the key lemma of Phillips for the optimal regularity of the second-order case, which illustrates the basic principles of the proof without technical difficulties that arise in the fractional order case. In chapter 3, we discuss basic existence results for our energy minimizers, and review certain classical results that we will use extensively. Subsequently, in chapter 4, we will prove the optimal regularity, and in chapter 5, the nondegeneracy properties of the minimizers.

Chapter 2

Optimal regularity for the 2nd order case

The optimal regularity for the problem in the 2nd order case was first obtained by Phillips [19]. Our method for proving the optimal regularity of the fractional case can be adapted to give an alternative proof for the 2nd order case. The main intuition behind our proof in the 2nd order case is free of certain technical issues that occur in the fractional case, and so we present it here first.

We obtain the optimal regularity of the energy minimizer u to the energy functional

$$J(u) = \int \frac{|\nabla u|^2}{2} + u^\gamma dx,$$

showing that $u \in C^{1,\beta-1}$ where $\beta = \frac{2}{2-\gamma}$, the scaling factor obtained by a calculation like that in §3.2. As with the fractional case, we assume the boundary data is non-negative, which allows us to assume the same for u. Notice that the Euler-Lagrange equations tell us that, when $u > 0$, u satisfies

$$\Delta u = \gamma u^{\gamma-1}.$$

As with Phillips, we seek to prove

Lemma 2.0.2. *There exists a constant $c_0(n)$ such that if*

$$\fint_{B_r} u dS > c_0 r^\beta,$$

then $u > 0$ *inside* B_r.

Since the scaling $u_\lambda(x) = \frac{1}{\lambda^\beta} u(\lambda x)$ preserves minimizers, we need only show this for $r = 1$, and scaling would take care of the rest.

Our proof works by showing that, when the average on the boundary is sufficiently large, a subsolution, or lower barrier to the energy minimizer, can be constructed which is wholly positive in the interior of B_r. There are two main stages to the proof: first, we detail what it means to be a subsolution, second, we construct a subsolution with the desired properties.

2.1 Subsolutions

We say that a function w is a subsolution, or lower barrier, to the energy minimizer, if w satisfies

$$\Delta w \geq M w^{\gamma-1}$$

whenever $w > 0$ inside B_r, setting $w = u$ along ∂B_r for Dirichlet boundary conditions, with M a large constant to be determined later. This terminology is natural because, as we shall see, $u \geq w$ inside B_r.

Let $v = \max(u, w)$, and consider the difference of energies given by

$$J(u) - J(v) = \int \frac{1}{2} \left((\nabla u)^2 - (\nabla v)^2 \right) + u^\gamma - v^\gamma dx.$$

Since u is the energy minimizer, we require that $J(u) - J(v) \leq 0$. However, we know that

$$\begin{aligned}
\frac{1}{2}\int (\nabla u)^2-(\nabla v)^2 dx &= \frac{1}{2}\int \nabla(u+v)\cdot\nabla(u-v)dx \\
&= \frac{1}{2}\int (v-u)\Delta(u+v)dx \\
&\geq \frac{1}{2}\int_{v>u>0}(v-u)(\gamma u^{\gamma-1}+Mv^{\gamma-1})dx+\frac{1}{2}\int_{v>u=0} Mv^{\gamma}dx.
\end{aligned}$$

We compare

$$\psi(s)=s^{\gamma}-t^{\gamma}$$

with

$$\phi(s)=\frac{1}{2}(s-t)(\gamma t^{\gamma-1}+Ms^{\gamma-1})$$

for $s>t\geq 0$. Clearly, $\phi(t)=\psi(t)=0$, and a bit of calculation assures us that $\phi'(s)\geq\psi'(s)$ for all s if we set $M\geq 2$. Thus,

$$\begin{aligned}
J(u)-J(v) &= \int \frac{1}{2}\left((\nabla u)^2-(\nabla v)^2\right)+u^{\gamma}-v^{\gamma}dx \\
&\geq \frac{1}{2}\int_{v>u>0}(v-u)(\gamma u^{\gamma-1}+Mv^{\gamma-1})dx+\frac{1}{2}\int_{v>u=0} Mv^{\gamma}dx \\
&+ \int u^{\gamma}-v^{\gamma}dx \\
&\geq 0,
\end{aligned}$$

with equality holding in the last statement only if $v\equiv u$. Hence, $u\geq w$, with $M=2$.

2.2 Construction of a positive subsolution

Our goal is to create a positive function w on the unit ball, such that $\Delta w\geq 2w^{\gamma-1}$. We will define w in three parts:

$$w(x)=w_1(x)+w_2(x)+w_3(x).$$

Let $\eta(x)$ be a radial, non-negative C^∞ function satisfying $\eta \leq 1$ everywhere, $\eta = 1$ when $|x| > \frac{3}{4}$, $\eta = 0$ when $|x| < \frac{1}{2}$, and $|\Delta\eta| \leq C'$ and $|\nabla\eta| \leq C'$ for some constant C'. We define

$$w_1(x) = \lambda\left(\eta(x)(1 - |x|)^\beta + (1 - \eta(x))\right),$$

and we claim that when $|x| > \frac{7}{8}$, we have

$$\Delta w_1 \geq 2w_1^{\gamma-1}$$

for the correct choice of λ.

In the region in question, it is easy to see that

$$\Delta w_1(x) = \beta(\beta - 1)(1 - r)^{\beta-2} - (n - 1)\beta\frac{(1 - r)^{\beta-1}}{r}.$$

We take the ratio of Δw_1 with w_1 in the region under concern, and we see that

$$\frac{\Delta w_1}{w_1^{\gamma-1}} = \beta\lambda^{2-\gamma}\left((\beta - 1)\frac{1 - r}{r})\right).$$

whence it is clear that a sufficiently large value of λ will suffice.

We set

$$w_2(x) = \mu(|x|^2 - 1),$$

where we pick μ sufficiently large so that $\Delta w_2 > -\Delta w_1 + 1$ everywhere inside $B_{\frac{7}{8}}$. It is clear from our design that $-\Delta w_1$ is bounded inside the region in question.

Finally, we let w_3 be the function which is harmonic inside B_1, with the same boundary values as the minimizer u along ∂B_1.

We claim that when $⨍_{\partial B_1} u dS$ is sufficiently large

1. $1 \geq 2w^{\gamma-1}$ on $B_{\frac{7}{8}}$.

2. $w \geq 0$ everywhere on B_1.

The Harnack inequality tells us that, on $B_{\frac{7}{8}}$, we have

$$w_3(x) \geq C \fint_{\partial B_1} udS.$$

To prove the first, it suffices if $\fint_{\partial B_1} udS$ is so large compared to μ so that $2\mu n > (C \fint_{\partial B_1} udS)^{\gamma-1}$. To prove the second, we bound w_3 from below by a suitably scaled truncated fundamental solution, and then

$$w_3(x) \geq \frac{C \fint_{\partial B_1} udS}{(\frac{8}{7})^{n-2} - 1} \left(\frac{1}{|x|^{n-2}} - 1 \right) \geq \mu(1 - |x|^2)$$

when $\fint_{\partial B_1} udS$ is sufficiently large.

Hence, we have $w \geq 0$ everywhere, and on $B_{\frac{7}{8}}$, we have

$$\Delta w \geq 1 \geq 2w^{\gamma-1}$$

while on $B_1 \setminus B_{\frac{7}{8}}$ we have

$$\Delta w \geq \Delta w_1 \geq 2w_1^{\gamma-1} \geq 2w^{\gamma-1},$$

and we are done.

Chapter 3

Preliminary considerations

In this chapter we identify some technical points of interest. First, we prove that minimizers of the energy exist. Second, we identify the scaling associated with the problem. Third, we list certain properties of the equation and minimizer that are known and will prove useful to our analysis.

3.1 Existence considerations and notation

We consider, in $B_+ = B_1 \cap \{y \geq 0\}$, minimizers of the energy

$$J(u) = \frac{1}{2}\int_{B_+} y^a |\nabla u|^2 dxdy + \int_{B_1\cap\{y=0\}} u^\gamma dx$$

in the space $H^1(B_+, a)$, with seminorm given by $||u|| = \int_{B_+} y^a(|\nabla u|^2)dxdy$, subject to the requirement $u \geq 0$. We impose non-negative Dirichlet conditions on $\partial B_1 \cap \{y > 0\}$, where $a = 1 - 2\sigma$, $0 < \sigma < 1$, $0 < \gamma < 1$. For sake of convenience, we denote $\Gamma = B_1 \cap \{y = 0\}$.

The energy can be interpreted as an averaging term which "lifts" the solution towards the boundary conditions, and a term which punishes u for being nonzero at $y = 0$, causing it to "stick." The set $\{u = 0\}$, which necessarily lies in $\{y = 0\}$, is called the *contact set* of u. The interface between $\{u = 0\} \cap \{y = 0\}$ and $\{u > 0\} \cap \{y = 0\}$ is called the *free boundary*.

Existence of minimizers is assured by the usual methods: consider a minimizing sequence for the energy. The first term of the energy is lower semicontinuous with respect to the norm for the usual reasons. The second term is continuous with respect to the norm for $L^2(\Gamma)$. From the extension result of Caffarelli-Silvestre we know that the trace of functions lying in $H^1(B_+, a)$ lie in $H^\sigma(\Gamma)$ [7], whence we apply the usual Sobolev embedding of H^σ inside L^2.

We will use $X = (x, y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}_+$.

3.2 Scaling of the problem

We seek the scaling that preserves minimizers of the energy

$$J(u) = \frac{1}{2}\int_{B_+} y^a |\nabla u|^2 dxdy + \int_\Gamma u^\gamma dx.$$

We consider the scaling

$$w(x, y) = \frac{1}{\lambda^\beta} u(\lambda x, \lambda y).$$

We find that

$$J(w) = \frac{1}{2}\int_\Omega y^a \lambda^{2-2\beta} |\nabla u(\lambda x, \lambda y)|^2 dxdy + \int_\Gamma \lambda^{-\beta\gamma} u^\gamma dx$$

which, after the change of variable, scales to

$$\lambda^{-a-n+2-2\beta}\frac{1}{2}\int_{\lambda\Omega} y^a |\nabla u|^2 dxdy + \lambda^{-\beta\gamma-n+1}\int_{\lambda\Gamma} u^\gamma dx.$$

Setting the exponents equal, we find that

$$\beta = \frac{2\sigma}{2-\gamma}.$$

3.3 Existence of a nontrivial free boundary

The Euler-Lagrange equations for $J(u)$ tell us that, in a distributional sense, the minimizer u satisfies

$$\nabla \cdot (y^a \nabla u) = 0$$

in the interior of B_+, and

$$\lim_{y\to 0} y^a \partial_y u = \gamma u^{\gamma-1} \tag{3.1}$$

along Γ wherever $u > 0$. The behavior of u in the hyperplane $y = 0$ is of particular interest: in this set, u separates into two behaviors; either $u = 0$ (the zero or contact set), or $u > 0$ (the positivity set). The interface between the two sets in the topology of $\mathbb{R}^n$, is called the free boundary. When $u > 0$, the minimizer satisfies equation (3.1).

Proposition 3.3.1 (Existence of nontrivial minimizers)**.** *There exist nontrivial energy minimizers u to our problem with nontrivial free boundaries. To be precise, if* $\sup u|_{\partial B_1 \cap \{y>0\}}$ *is sufficiently small, then along* $y = 0$ *there is a nontrivial zero set.*

Proof. This will be a proof by contradiction. Suppose this is not true, that is to say, $\sup u|_{\partial B_1 \cap \{y>0\}} \leq \varepsilon$ but u is uniformly positive along Γ. Take a C^∞ positive test function ϕ with compact support contained within B_1. Then the Euler-Lagrange equations tell us that

$$\begin{aligned}
\int_\Gamma \phi \lim_{y\to 0} y^a \partial_y u dx &= \int_\Gamma \phi \gamma u^{\gamma-1} dx \\
&= \int_\Gamma u \lim_{y\to 0} y^a \partial_y \phi dx - \int_{B_1^+} u \nabla \cdot (y^a \nabla \phi) dx dy.
\end{aligned}$$

Then the term on the first line grows faster than $c\varepsilon^{\gamma-1}$ as $\varepsilon \to 0$, which is to say, it becomes very large, while the terms on the second line are of $O(\varepsilon)$ as $\varepsilon \to 0$. For ε sufficiently small, this is a contradiction. $\square$

3.4 Properties of the σ-Laplace operator

Caffarelli and Silvestre [7] showed that in the upper half space $\mathbb{R}^{n+1}_+$, the effective normal derivative operator is equivalent to the fractional Laplacian of order σ.

$$\lim_{y\to 0} y^a \partial_y u = -C(-\Delta)^\sigma u$$

(this is just Theorem 1.0.1 repeated here for convenience). In addition to its instantiation as a Neumann derivative-like operator for functions on the half-space, the fractional Laplacian is also an integrodifferential operator, with Fourier symbol $|\xi|^{2\sigma}$. There is a large body of work studying the properties of this operator (the sources for these properties are [20], chapter V of [21], [16]).

We list certain properties on which we will rely in the remainder of this section which may not be commonly known; first, we list some useful properties arising (or most easily seen) through the extension technique in §3.4.1. Subsequently, we list some others arising from direct examination of the fractional Laplace operator itself (either as a Fourier multiplier or as an integrodifferential operator) in §3.4.2.

3.4.1 Properties arising from equations with A_2 weights

Fabes, Jerison, Kenig, and Serapioni ([13],[11],[12]) extended the De Giorgi-Nash-Moser theory of divergence-form elliptic equations to degenerate elliptic equations with Muckehnhoupt A_2 weights; these are equations of the form

$$\nabla \cdot (A(X)\nabla u) = 0$$

where the matrix A satisfies

$$\lambda w(X)|\xi|^2 \leq \sum A_{ij}(X)\xi_i\xi_j \leq \Lambda w(x)|\xi|^2$$

for $\xi \in \mathbb{R}^n$, with weight functions w satisfying

$$\left(\frac{1}{|B|}\int_B w(X)dx\right)\left(\frac{1}{|B|}\int \frac{1}{w(X)}dX\right) \leq C$$

for all balls B. These results apply to our problem, since if we let $X = (x, y)$ with $w(x, y) = y^{1-2\sigma}$, then w is an A_2 weight. In particular, such properties as the strong maximum principle, interior Holder regularity of solutions, and the Harnack inequality for nonnegative solutions all hold.

There are certain other properties, such as the De Giorgi Oscillation Lemma and a more specific form of the De Giorgi-Nash-Moser Harnack inequality, which follow directly from their work but are not explicitly stated. A discussion of those properties follows.

For purposes of notation for this section, the equation satisfied is

$$\nabla \cdot (A\nabla u) = 0$$

where $w(X)\lambda|\xi|^2 \leq \xi^T A\xi \leq w(X)\Lambda|\xi|^2$ and w is some A_2 weight.

3.4.1.1 The De Giorgi oscillation lemma

Lemma 3.4.1. *Suppose u is a positive supersolution in B_2 with*

$$|\{X \in B_1; u \geq 1\}| \geq \varepsilon |B_1|.$$

Then there exists a constant C depending only on ε, n, and σ such that

$$\inf_{B_{\frac{1}{2}}} u \geq C.$$

Although we prove the former statement, the obvious corollary concerning subsolutions follows from applying the lemma to $1 - u$, and is what we actually use:

Corollary 3.4.2. *Suppose u is a subsolution in B_2 with*

$$|\{X \in B_1; u \leq 0\}| \geq \varepsilon |B_1|$$

and $u \leq 1$. Then there exists a constant $0 < \mu < 1$ depending only on ε, n, and σ such that

$$\sup_{B_{\frac{1}{2}}} u \leq \mu.$$

In line with [13], we let $w(B) = \int_B y^a dxdy$ represent the integral of our weight over a ball. The proof of this lemma depends on a Poincare inequality:

Lemma 3.4.3. *For any $\varepsilon > 0$ there exists a $C(\varepsilon, \sigma)$ such that for $u \in H^1(B_1)$ with*

$$|\{X \in B_1; u = 0\}| \geq \varepsilon |B_1|$$

we have

$$\int_{B_1} y^a u^2 dxdy \leq C \int_{B_1} y^a |\nabla u|^2 dxdy.$$

Proof. It is a classical result (see, for example, Kinderlehrer and Stampacchia, II.A.15 [15]) that for smooth functions u on B_r that vanish on a set of measure at least εB_r, we have

$$|u(X)| \leq C \int_{B_r} \frac{|\nabla u(z)|}{|X - z|^{n-1}} dz.$$

However, this is precisely the point of departure for Fabes, Kenig, and Serapioni (Theorem 1.2) [13], where they prove for functions u satisfying this condition, we have

$$\int_{B_r} y^a u^2 dxdy \leq Cr \int_{B_r} y^a |\nabla u|^2 dxdy$$

which is precisely the result we were looking for. $\square$

The rest of the proof follows the proof given in [14], which is reasonably short, so we reproduce it here.

Proof. Assume $u \geq \delta > 0$ - we will see that the final result is insensitive to δ, and so we can let $\delta \to 0+$ at the end.

Let $v = (\log u)^-$, then v is a subsolution to the equation, bounded by $\log \delta^{-1}$. Then we have (Theorem 2.3.1 in [13])

$$\sup_{B_{\frac{1}{2}}} v \leq C \left(\frac{1}{w(B_1)} \int_{B_1} y^a v^2 dxdy \right)^{\frac{1}{2}}.$$

Applying the Poincare inequality, we see that

$$\sup_{B_{\frac{1}{2}}} v \leq C \left(\frac{1}{w(B_1)} \int_{B_1} y^a |\nabla v|^2 dxdy \right)^{\frac{1}{2}}.$$

We set the test function $\phi = \frac{\zeta^2}{u}$ for $\zeta \in C_0^1(B_2)$. Then we obtain

$$0 \leq \int y^a \nabla u \cdot \nabla(\frac{\zeta^2}{u}) dydx = -\int \frac{\zeta^2}{u^2}(\nabla u)^2 + 2\frac{\zeta}{u}\nabla u \cdot \nabla\zeta dxdy,$$

whence we obtain

$$\int y^a \zeta^2 |\nabla(\log u)|^2 dxdy \leq C \int y^a |\nabla\zeta|^2 dxdy.$$

By fixing $\zeta = 1$ on B_1 and giving it bounded first derivative, we have

$$\int_{B_1} y^a |\nabla(\log u)|^2 dxdy \leq Cw(B_2).$$

Combining our statements, we find

$$\sup_{B_{\frac{1}{2}}} (\log u)^- \leq C\left(\frac{w(B_2)}{w(B_1)}\right)^{\frac{1}{2}},$$

which gives

$$\inf_{B_{\frac{1}{2}}} u \geq e^{-C\left(\frac{w(B_2)}{w(B_1)}\right)^{\frac{1}{2}}}.$$

$\frac{w(B_2)}{w(B_1)}$ is bounded since all A_p weights have a doubling property (see [22], V.1.5), and hence

$$\inf_{B_{\frac{1}{2}}} u \geq C.$$

□

3.4.1.2 The De Giorgi-Nash-Moser Harnack Inequality

Theorem 3.4.4 (DeGiorgi-Nash-Moser Interior Harnack Inequality)**.** *Let u be a non-negative solution in B_1 to the equation. Then for $r < 1$, we have*

$$\sup_{B_r} u \leq c(1-r)^{-p} \inf_{B_r} u,$$

where c, p do not depend on r or the center of the ball.

Proof. This fact is a straightforward extension of the standard interior Harnack inequality (proved in [13]), which simply states that, so long as the equation is satisfied in B_2, we have

$$\sup_{B_{\frac{1}{2}}} u \leq C \inf_{B_{\frac{1}{2}}} u,$$

where $C > 1$ is invariant under translation or dilation of the ball. In what follows, we assume $r > \frac{1}{2}$, since the standard inequality proves the result for the case $r \leq \frac{1}{2}$, and that the balls are closed.

Suppose $\frac{1}{2} > r > \frac{1}{4}$. Consider the collection of balls $B_{\frac{1}{2}}(X)$, where $X \in \partial B_{\frac{1}{2}}$. The union of these balls, along with $B_{\frac{1}{2}}(0)$, is precisely $B_{\frac{3}{4}}$. For every $X \in \partial B_{\frac{1}{2}}$, we have

$$\sup_{B_{\frac{1}{4}}(X)} u \leq Cu(X) \leq C \sup_{B_{\frac{1}{2}}} u.$$

Let $X^* \in B_{\frac{1}{2}}$ be such that $B_{\frac{1}{4}}(X^*)$ is a ball containing $\inf_{B_{\frac{3}{4}}} u$. Notice that

$$C \sup_{B_{\frac{1}{2}}} u \leq C^2 u(X^*) \leq C^3 \inf_{B_{\frac{3}{4}}} u.$$

Hence, we have

$$\sup_{B_{\frac{3}{4}}} u \leq C^3 \inf_{B_{\frac{3}{4}}} u.$$

We use the same argument to extend from $B_{1-\frac{1}{2^k}}$ to $B_{1-\frac{1}{2^{k+1}}}$ inductively, and we get

$$\sup_{B_{1-2^{-k}}} u \leq C^{2k-1} \inf_{B_{1-2^{-k}}} u$$

and so on, until we reach the first k such that $1-2^{-k} > r$. At this point, we recognize that $k \approx -\log(1-r)$. Plugging in, we get the desired result. □

3.4.2 Other properties

We will also use the fact that the operator given by $Lu = \nabla \cdot (y^a \nabla u)$ has no explicit dependence on the x-variable. Hence, solutions to $\nabla \cdot (y^a \nabla u) = 0$ are not only C^α, but that regularity can be bootstrapped to provide estimates on the x derivatives of u of all orders, a fact proved in [9]:

Proposition 3.4.5. *Suppose* $\nabla \cdot (y^a \nabla u) = 0$ *in* $B_r(X_0)$. *Then we have*

$$\sup_{B_{\frac{r}{2}}(X_0)} |D_x^k u| \leq \frac{C}{r^k} \operatorname*{osc}_{B_r(X_0)} u,$$

where osc *is taken to mean* $\sup u - \inf u$. *Furthermore,*

$$[D_x^k u]_{C^\alpha(B_r(X_0))} \leq \frac{C}{r^{k+\alpha}} \operatorname*{osc}_{B_r(X_0)} u,$$

where D_x^k *means any kth order derivative lying purely in the x directions, and* $[f]_{C^\alpha}$ *denotes the* α*-Holder seminorm of* f.

By even reflection of u across the hyperplane $y = 0$, this result can also be very useful when $\lim_{y\to 0} y^a \partial_y u = 0$ and $\nabla \cdot (y^a \nabla u) = 0$ in $B_r(x, 0) \cap \{y > 0\}$.

A related estimate when $\lim_{y\to 0} y^a \partial_y u = -C_{n,\sigma}(-\Delta)^\sigma u \neq 0$ was derived by Silvestre in [20] from directly examining the fractional Laplacian:

Proposition 3.4.6. *Let* $w = (-\Delta)^\sigma u$. *Assume* $w, u \in L^\infty(\mathbb{R}^n)$. *Then, for* $\sigma > 0$, *if* $2\sigma \leq 1$, *then for any* $\alpha < 2\sigma$, *we have*

$$||u||_{C^\alpha(\mathbb{R}^n)} \leq C(||u||_{L^\infty(\mathbb{R}^n)} + ||w||_{L^\infty(\mathbb{R}^n)}).$$

If $2\sigma > 1$, *then the same result holds for the* $C^{1,\alpha-1}$ *Holder norm.*

It is often desireable, given boundary data on $\mathbb{R}^n \times \{0\}$, to find a function u which satisfies $\nabla \cdot (y^a \nabla u) = 0$ in $\mathbb{R}^{n+1}_+$. This can be done by means of convolution with the following Poisson kernel (Caffarelli-Silvestre):

Proposition 3.4.7 (Poisson kernel for the extension [7])**.** *Let*

$$P(x,y) = C_{n,\sigma} \frac{y^{2\sigma}}{(|x|^2 + y^2)^{\frac{n+2\sigma}{2}}}.$$

Then if $f(x)$ *is a function on* $\mathbb{R}^n$*,*

$$u(x,y) = \int_{\mathbb{R}^n} f(x')P(x-x',y)dx'$$

is a function satisfying $\nabla \cdot (y^a \nabla u) = 0$ *on* $\mathbb{R}^{n+1}_+$.

The generalized Hopf lemma for σ-harmonic functions in $\mathbb{R}^n$ (first stated in [5]) is useful:

Lemma 3.4.8 (Generalized Hopf Lemma for σ-harmonic functions)**.** *If a smooth function* $v(x)$ *satisfies* $(-\Delta)^\sigma v = 0$ *in some smooth domain* $\Omega \subset \mathbb{R}^n$*,* v *being non-negative in general, and positive in the interior of* Ω*, and if there exists a point* $x_0 \in \partial\Omega$ *where* $v(x) = 0$*, then there exists* C *such that* $v(x) \geq C((x-x_0)\cdot \nu(x_0))^\sigma$ *where* $\nu(x_0)$ *is the interior normal to* $\partial\Omega$ *at* x_0.

Chapter 4

Optimal regularity

The goal of this chapter is to obtain the optimal regularity of energy minimizers u. In particular, we seek to show that u grows away from the free boundary like a power of the distance. To be precise, $u(X) \leq Cd^{\beta}$, where d is the distance of X to the free boundary, and $\beta = \frac{2\sigma}{2-\gamma}$ is the scaling factor obtained in §3.2.

As corollaries, we obtain some regularity results: restricted to $\{y = 0\}$, u lies in the Holder space C^{β}. If $\beta > 1$, we will prove that $u \in C^{1,\beta-1}$, which by abuse of notation we will still refer to as C^{β}. In the interior domain where $y > 0$, we still obtain $u \in C^{\beta}$ when $\beta < 1$. When $\beta \geq 1$, we find that $u \in C^{\alpha}$ for any $\alpha < 1$.

To obtain optimal growth, we consider a point p_0 which is at some distance (normalized to 1) from the nearest free boundary point, which we will take to be 0. We will use a variant of the boundary Harnack inequality due to Caffarelli to compare the value of $u(p_0)$ with some point p_1 in the interior of B_+, specifically, showing that $u(p_1) \geq Mu(p_0)$. We can then use the regular Harnack inequality in the interior to show that, in a smaller ball about the free boundary point, the boundary values are controlled by $u(p_1)$, and hence by $u(p_0)$. We then prove that, if the boundary values in the upper half ball are too large at the right scale, then $u(0)$ is strictly positive, meaning that $u(p_0)$ cannot be too large. Subsequently, rescaling obtains the desired regularity.

Our main tools to prove optimal growth are a variant of the Boundary Harnack Inequality[1], and a lemma stating that if the values along the boundary of the upper half-sphere are sufficiently large, then the minimizer of the energy taking boundary conditions along the sphere has a positive value at the center.

Theorem 4.0.9 (Variant Boundary Harnack Inequality)**.** *Let u be a non-negative solution of the equation $\nabla \cdot (y^a \nabla u) = 0$ in B^+, which satisfies $\lim_{y\to 0} y^a \partial_y u \geq 0$ along $\{y = 0\}$, taking on some continuous boundary values along $\{y = 0\}$, with $u(0, \frac{1}{4}) = 1$. Then inside $B^+_{\frac{1}{2}}$, we have $u \leq M$ for some constant $M(n, \sigma)$.*

Lemma 4.0.10 (Minimizers with large averages are positive at the center)**.** *Let u be a minimizer of the energy $J(u)$ inside $B_r \cap \{y > 0\}$, taking non-negative boundary values along $\partial B_r \cap \{y > 0\}$. $\exists c_0 > 0$ such that, if*

$$u|_{\partial B \cap \{y > \frac{r}{2}\}} \geq c_0,$$

then we have

$$u(x, y) > 0 \forall (x, y) \in B_{\frac{r}{3}} \cap \{y \geq 0\},$$

and in fact there exists a constant c such that

$$u(x, y) \geq c \forall (x, y) \in B_{\frac{r}{6}} \cap \{y \geq 0\}.$$

[1] There are two types of results which are, confusingly, both called the Boundary Harnack Inequality in the literature. In addition to the result here, which states that values in the neighborhood of the boundary are uniformly bounded in terms of the value at an interior point, there is a closely related result which states that for two solutions which are both 0 along a stretch of the boundary, their ratios are locally Holder-continuous. We follow the naming convention of Caffarelli and Salsa [8] and call the first result the Boundary Harnack Inequality, and the second result the Boundary Comparison Principle.

Together, these suffice to prove our result, namely:

Theorem 4.0.11. *There exists a constant K such that in any ball of radius r centered at a point x_0 where $u(x_0) = 0$, we have*

$$|u(x) - u(x_0)| \leq K|x - x_0|^\beta$$

for all $x \in B_r(x_0) \cap \Gamma$ where $\beta = \frac{2\sigma}{2-\gamma}$.

Proof. Without loss of generality, let 0 be a point such that $u(0) = 0$, and X_* be a point such that $|X_*| = 1$. We claim that $u(X_*) \leq K$. Suppose this is not true, that is to say, we can make $u(X_*)$ as large as we wish. Then by the variant boundary Harnack inequality applied to $B_2(0)$ we have

$$u(0, 1) \geq \frac{u(X_*)}{M},$$

where M is the constant from the variant boundary Harnack inequality. By applying the DeGiorgi-Nash-Moser Harnack inequality to u about we discover that we have

$$u(x, y) \geq Cu(X_*)y,$$

whence, by invoking Lemma 4.0.10, we have $u(0) > 0$, a contradiction on our original assumption. Thus, there exists a constant K such that $u(X_*) \leq K$, as desired. By rescaling the problem, we recover our desired result. $\square$

Corollary 4.0.12. *Let u be an energy minimizer in a subset of $\mathbb{R}^{n+1}_+$ containing B_1^+, with 0 a free boundary point. Then considered as a function along the set $\{y = 0\}$, u is a C^β function, with $||u||_{C^\beta(B_{\frac{1}{2}})} \leq C$, where C depends only on σ, γ, and n.*

Corollary 4.0.13. *Let u be an energy minimizer in a subset of* $\mathbb{R}^{n+1}_+$ *containing* B_1^+ B_1^+*, with 0 a free boundary point. Then in* $B^+_{\frac{1}{2}}$*, u is a* C^β *function, with* $\|u\|_{C^\beta(B^+_{\frac{1}{2}})} \leq C$*, where C depends only on* σ, γ*, and n, if* $\beta < 1$*. If* $\beta \leq 1$*, u is a* C^α *function for any* $\alpha < 1$*, with*

$$\|u\|_{C^\alpha(B^+_{\frac{1}{2}}} \leq C(\sigma, \gamma, n, \alpha).$$

The proof of these statements, and a discussion of the C^β norm estimates, are covered in 4.3.

4.1 Variant Boundary Harnack Inequality

The proof of this variant of the boundary Harnack inequality follows the same lines as the standard proof of the boundary Harnack inequality provided by Caffarelli et alia [4]. The proof uses two classical facts from the De Giorgi-Nash-Moser theory, which was extended to the theory of degenerate elliptic equations with A_2 weights by Fabes, Kenig, and Serapioni [13], a class that includes the equation $\nabla \cdot (y^a \nabla v) = 0$.

The first fact is the De Giorgi-Nash-Moser Harnack inequality (Theorem 3.4.4), which states that for a non-negative solution in B_1,

$$\sup_{B_r} u \leq c(1-r)^{-p} \inf_{B_r} u$$

where $p > 0$.

The second fact is the De Giorgi oscillation lemma (Lemma 3.4.1), which says that a subsolution v in B_1 satisfying

- $v \leq 1$.
- $|\{v \leq 0\}| = a > 0$.

has the property that

$$\sup_{B_{1/2}} v \leq \mu(a) < 1.$$

We proceed by contradiction. Let $u(0, \frac{1}{2}) = 1$, and extend u over the line $y = 0$ by even reflection. Suppose there is no M which can bound values of u inside the ball. Then u achieves its maximum in it $M_0 > M$, at some point $X_0 = (x_0, y_0)$. The Harnack inequality tells us that the distance to the boundary, y_0, satisfies

$$y_0 \leq d_0 = \left(\frac{c}{M}\right)^{\frac{1}{p}}.$$

We now proceed with a construction we repeat for each successive value of n, starting with $n = 0$:

Consider now $B_{Kd_n}(x_n, 0)$ (the projection of X_n to the plane $y = 0$), for K large, greater than, say, 4. For points satisfying $y > 2d_n$, we have, by the Harnack inequality, that

$$u(X) \leq c(2d_n)^{-p} = \frac{M_n}{2^p}.$$

The set $\{y > 2d_n\}$ has measure at least a fixed fraction of $B_{Kd_n}(x_n, 0)$, independent of K. Thus, if we let $M_{n+1} = \sup_{B_{Kd_n}(x_n,0)} u$, we see that inside, say, $B_{2d_n}(x_n, 0)$, by the oscillation lemma, we have that

$$M_n(1 - 2^{-p}) \leq \sup_{B_{2d_n}(x_n,0)} (u - \frac{M_n}{2^p}) \leq \mu(K)(M_{n+1} - \frac{M_n}{2^p})$$

with $\mu \to 0$ as K becomes large. Thus

$$M_{n+1} > M_n(2^{-p} + \frac{1-2^{-p}}{\mu(K)}).$$

We pick K sufficiently large that the factor on the right hand side is some fixed positive $\lambda > 1$. We let X_{n+1} be the point where $u(X_{n+1}) = M_{n+1}$ inside $B_{Kd_n}(x_n, 0)$.

Thus we have a sequence of points X_n. Notice that K does not change, and hence neither does λ. As $n \to \infty$, we have

$$u(X_n) \geq \lambda^n M_0 \to \infty$$

while

$$y_n \leq d_n = (cM_0)^{-\frac{1}{p}} \lambda^{-\frac{n}{p}} \to 0.$$

The distances between the points satisfy

$$|X_{n+1} - X_n| \leq K d_n$$

and so the sequence has

$$d(X_0, X_n) \leq K \sum d_n \leq K \frac{(cM)^{-\frac{1}{p}}}{1-\lambda^{-\frac{1}{p}}},$$

which can be made to converge inside $B_{\frac{9}{16}}$ for M sufficiently large, giving us a sequence of points X_n, with limit points where u blows up along $y = 0$. This contradicts our original assumption that u continuously assumes values along the boundary $\{y = 0\}$.

4.2 The center is positive when the boundary is large

Our main lemma (Lemma 4.0.10) consists of demonstrating that when c_0 is sufficiently large, a subsolution which is purely positive in $B_{\frac{1}{3}}$ can be built, which serves as a lower barrier to the solution.

4.2.1 Conditions for a subsolution

We seek sufficient conditions for a function to be a *subsolution* of our variational problem. One way to do this is to show that, for a subsolution w, where $u > w$, we can improve the energy: if $v = \max(u, w)$, then

$$J(u) - J(v) = \frac{1}{2}\int y^a \left(|\nabla u|^2 - |\nabla v|^2\right) dydx + \int_\Gamma u^\gamma - v^\gamma dx \geq 0$$

since u is the energy minimizer. Clearly, the second term is negative; our approach lies on setting conditions so that the first term dominates the second.

We assume u, v sharing the same Dirichlet boundary conditions along ∂B, and integrate by parts:

$$\begin{aligned}\int \frac{1}{2} y^a(|\nabla u|^2 - |\nabla v|^2)dxdy &= \frac{1}{2}\int y^a(\nabla u + \nabla v)\cdot(\nabla u - \nabla v)dxdy \\ &= -\frac{1}{2}\int (u - v)\nabla\cdot(y^a\nabla(u + v))dxdy \\ &\quad -\frac{1}{2}\int_\Gamma (u - v)\lim_{y\to 0} y^a\partial_y(u + v)dx.\end{aligned}$$

We define $v = \max(u, w)$, where w satisfies

$$\lim_{y\to 0} y^a\partial_y w \geq Mw^{\gamma-1}$$

along Γ, and $\nabla \cdot (y^a \nabla w) = 0$ in B_1. Then $\lim_{y \to 0} \partial_y v \geq M v^{\gamma-1}$ on those portions where $v > u$, whence we can write

$$\int y^a (|\nabla u|^2 - |\nabla v|^2) dx dy \geq \int_{\Gamma \cap \{v > u > 0\}} (v - u)(\gamma u^{\gamma-1} + M v^{\gamma-1}) dx + \int_{\Gamma \cap \{v > u = 0\}} M v^\gamma dx.$$

Recall now that

$$J(u) - J(v) = \frac{1}{2} \int_B y^a (|\nabla u|^2 - |\nabla v|^2) dx dy + \int_\Gamma u^\gamma - v^\gamma dx.$$

Since u is the energy minimizer, we need for this term to be negative.

We consider the functions

$$\psi(s) = s^\gamma - t^\gamma,$$

and

$$\phi(s) = \frac{1}{2}(s - t)(\gamma t^{\gamma-1} + M s^{\gamma-1}).$$

Clearly, $\phi(t) = \psi(t) = 0$. We now examine their behavior in the range $0 \leq t < s$. When $s > t$,

$$\psi'(s) = \gamma s^{\gamma-1} \leq \phi'(s) = \frac{1}{2}\gamma t^{\gamma-1} + \gamma M s^{\gamma-1} + \gamma(1 - \gamma) M t s^{\gamma-2}.$$

Thus, $\phi(s) > \psi(s)$ when $s > t$, and we can write

$$\begin{aligned}
J(u) - J(v) &= \frac{1}{2} \int_B y^a (|\nabla u|^2 - |\nabla v|^2) dx dy + \int_\Gamma u^\gamma - v^\gamma dx \\
&\geq \frac{1}{2} \int_{\Gamma \cap \{v > u > 0\}} (v - u)(\gamma u^{\gamma-1} + M v^{\gamma-1}) dx \\
&\quad + \frac{1}{2} \int_{\Gamma \cap \{v > u = 0\}} M v^\gamma dx + \int_\Gamma u^\gamma - v^\gamma dx \\
&\geq 0,
\end{aligned}$$

with the last equality being strict if v differs from u on a set with positive measure. This is satisfied if we set $M = 2$.

Hence, whenever such a w exists, we can decrease the energy of u, a contradiction on the definition of u as the energy minimizer. The construction of such a w, which is positive on $B_{\frac{1}{3}}$ and greater than a fixed constant on $B_{\frac{1}{6}}$, is thus sufficient to prove Lemma 4.0.10.

4.2.2 Construction of such a subsolution

We want our subsolution w to have three properties: we would like our w to take the same values as u along $\partial B \cap \{y > 0\}$, we would like it to satisfy the conditions

$$\nabla \cdot (y^a \nabla w) = 0$$

in B_+, and along Γ we would like

$$\lim_{y \to 0} y^a \partial_y w \geq 2w^{\gamma-1}$$

wherever $w > 0$, and finally we want $w > 0$ in $B_{\frac{1}{3}} \cap \Gamma$. We will define our w in two parts.

$$w = w_1 + w_2.$$

We set w_1 by setting, for $x \in \mathbb{R}^n$,

$$\psi(x) = \begin{cases} 0 & |x| > \frac{1}{3} \\ -(1 - 3|x|)^{\beta - 2\sigma} & |x| \leq \frac{1}{3} \end{cases}.$$

Let

$$(I_{2\sigma}\psi)(x) = C_{n,\sigma} \int \frac{\psi(z)}{|x - z|^{n-2\sigma}} dz$$

be the Riesz potential of ψ. We need a technical lemma relying on classical results in the theory of fractional integration and Riesz potentials, whose proof we defer to §4.2.3:

Lemma 4.2.1. $(I_{2\sigma}\psi)(x)$ *is well defined and continuous as a function, radial, has fractional Laplacian equal to* $\psi(x)$*, and furthermore, there exists* $\delta > 0$ *such that*

$$|(I_{2\sigma}\psi)(r) - (I_{2\sigma}\psi)(\frac{1}{3})| \leq C(1-3r)^{\alpha}$$

for $\frac{1}{3} > r = |x| > \frac{1}{3} - \delta$ *where* $\min(\beta, 1) > \alpha > \sigma$*.*

We let $b(x)$ be equal to $I_{2\sigma}\psi$ on $\mathbb{R}^n \setminus B_{\frac{1}{3}}$, and have $(-\Delta)^{\sigma} b = 0$ inside $B_{\frac{1}{3}}$, and then we set

$$\tilde{w}(x) = (I_{2\sigma}\psi)(x) - b(x).$$

b is the solution to the standard Dirichlet problem for the fractional Laplacian; its existence is guaranteed by the standard theory (see, e.g., Landkof [16]). Notice that $\tilde{w}$ is σ-subharmonic. This means it is negative inside $B_{\frac{1}{3}}$, and 0 outside of it. Furthermore, the maximum principle for σ-harmonic functions (Lemma 3.4.8) applied to $b(x)$ tells us that there is a constant such that

$$|b(x) - b(\frac{1}{3})| \leq C(1-3r)^{\sigma}.$$

Now we let

$$w_1(x, y) = C_{n,\sigma} \int \frac{y^{2\sigma}\tilde{w}(z)}{((x-z)^2 + y^2)^{\frac{n+2\sigma}{2}}} dz,$$

where z ranges over $\mathbb{R}^n$. This is, of course, the Poisson kernel for the fractional Laplacian convolved with $\tilde{w}$, giving us a w_1 that satisfies $\nabla \cdot (y^a \nabla w_1) = 0$ in the interior, which takes on the values of $\tilde{w}$ along $\{y = 0\}$, satisfying $\lim_{y\to 0} y^a \partial_y w_1(x, 0) = \psi(x)$ (by the extension result of Caffarelli and Silvestre [7]).

For the sake of future estimations, it is helpful to bound $-w_1$ from above by an auxiliary function. We let $q_0 = 2\sup(-w_1)$, and we let

$$q(x) = \begin{cases} q_0 & |x| < \frac{1}{3} - \delta \\ 0 & |x| > \frac{1}{3} \end{cases}.$$

and let q satisfy $(-\Delta)^\sigma q = 0$ on the annular ring $\frac{1}{3} - \delta < |x| < \frac{1}{3}$. The comparison principle for fractional-harmonic functions then tells us that $q \geq -w_1$ on Γ. We extend q to $\mathbb{R}^{n+1}_+$ in the usual way via the Poisson kernel:

$$Q(x, y) = C_{n,\sigma} \int \frac{y^{2\sigma} q(z)}{((x-z)^2 + y^2)^{\frac{n+2\sigma}{2}}} dz.$$

Thus, we have

Proposition 4.2.2. *$Q(x, y) \geq |w_1(x, y)|$ in the upper half-space $\mathbb{R}^{n+1}_+$.*

We set w_2 with boundary conditions

$$w_2(X) = \begin{cases} u - w_1(X) & X \in \partial B_1 \cap \{y > 0\} \\ 0 & X \in \Gamma \setminus B_{\frac{1}{3}} \end{cases},$$

and let it satisfy the problem

$$\nabla \cdot (y^a \nabla w_2) = 0$$

when $X \in B_+$, and

$$\lim_{y\to 0} y^a \partial_y w_2 = 0$$

when $X \in \Gamma \cap B_{\frac{1}{3}}$.

Now we need to estimate properties of $w = w_1 + w_2$, which we do by comparing w_2 to Q

Proposition 4.2.3. *For any $\lambda > 0$, a sufficiently large value of c_0 will ensure that $w_2(X) \geq \lambda Q(X)$ inside B_+*

Proof. Since both functions satisfy $\nabla \cdot y^a(\nabla v) = 0$ inside B_+, it suffices to examine their relative behavior along $\partial_{B_1} \cap \{y > 0\}$ and along Γ.

Along $\partial_{B_1} \cap \{y > 0\}$, the boundary comparison principle (see [12] for a proof in the case of A_2 weighted degenerate elliptic equations) tells us that

$$u(x, y) \geq Cc_0 y^{2\sigma}$$

since $u \geq c_0$ when $y > \frac{1}{2}$. We also have from the formula that $Q(x, y) \leq Cq_0 y^{2\sigma}$. Hence, we just need c_0 to be sufficiently large.

The behavior along Γ is a touch trickier. We divide our analysis of the behavior of w_2 along Γ into two parts: the first part concerns the interior of $B_{\frac{1}{3}-\delta}$, where δ is from Lemma 4.2.1, and the other in the thin annular ring $\frac{1}{3} - \delta < |x| < \frac{1}{3}$. Clearly, $\Gamma \setminus B_{\frac{1}{3}}$ is not taken care of, since w_2 and Q are identically 0 there.

We note that $w_2|_{\partial B \cap \{y>0\}} > 0$, so that we can apply the Harnack inequality in the interior. We bound w_2 from below by a function $\hat{w}_2$, which we define as follows: let

$$\hat{w}_2(X) = \begin{cases} u - w_1(X) & X \in \partial B_1 \cap \{y > 0\} \\ 0 & X \in \Gamma \end{cases}$$

and let it satisfy the problem

$$\nabla \cdot (y^a \nabla \hat{w}_2) = 0.$$

Clearly $0 \leq \hat{w}_2 \leq w_2$ in the domain. Since we know w_2 in $\{y > \frac{1}{2}\}$ is greater than c_0, it follows that so too $\hat{w}_2$ at interior points, such as, say, $X = (0, \frac{1}{6})$, is linear in c_0, and hence so is w_2. We apply the Harnack inequality to w_2 inside the ball $B_{\frac{1}{3}}$ to see that $w_2 \geq Cc_0$ in $B_{\frac{1}{3}-\delta}$ can be made as large as we wish, where δ is from lemma 4.2.1. Thus, inside $B_{\frac{1}{3}-\delta} \cap \Gamma$, we can choose c_0 so that $w_2 \geq Q$.

In the annular ring proper, both $Q(x, y)$ and $w_2(x, y)$ satisfy $\lim_{y\to 0} y^a \partial_y v = 0$, whence we can invoke the Hopf lemma to see that $w_2 \geq Q$. □

Corollary 4.2.4.

$$w = w_1 + w_2 \geq (\lambda - 1)Q,$$

and hence by making λ sufficiently large we can make $w \geq Cq_0$ in $B_{\frac{1}{6}}$.

We close our construction with a lemma, which shows that w has all the desired properties of a subsolution.

Lemma 4.2.5.

$$w_1 + w_2 \geq 0$$

and for λ from the previous proposition sufficiently large (which is really to say for c_0 sufficiently large), we have

$$\lim_{y\to 0} y^a \partial_y w \geq 2w^{\gamma-1}$$

along Γ, wherever $w \neq 0$.

Proof. Since $w_1 \geq -Q$ and $w_2 \geq \lambda Q$, we have

$$w = w_1 + w_2 \geq w_2 - Q \geq 0.$$

On $\Gamma \cap B_{\frac{1}{3}-\delta}$, we have

$$\begin{aligned} \lim_{y\to 0} y^a \partial_y w &= \lim_{y\to 0} y^a \partial_y w_1 \\ &= (1-3|x|)^{\beta-2\sigma} \\ &\geq (3\delta)^{\beta-2\sigma}. \end{aligned}$$

By setting λ sufficiently large, we can attain

$$\begin{aligned} (3\delta)^{\beta-2\sigma} &\geq 2(\lambda q_0)^{\gamma-1} \\ &\geq 2w_2^{\gamma-1} \\ &\geq 2w^{\gamma-1}. \end{aligned}$$

On the annular ring, we invoke Lemma 3.4.8 to see that there is a constant c such that

$$q(x) \geq c(1-3|x|)^{\sigma},$$

whence we derive the relation

$$w(x) \geq (\lambda - 1)c(1-3|x|)^{\sigma}.$$

By setting λ sufficiently large, we can make

$$2^{\frac{1}{\gamma-1}}(\lambda-1)c(1-3|x|)^{\sigma} \geq (1-3|x|)^{\beta}$$

in the entire annular ring, whence we can attain

$$\begin{aligned}\lim_{y\to 0} y^a \partial_y w &= \lim_{y\to 0} y^a \partial_y w_1 \\ &= (1-3|x|)^{\beta-2\sigma} \\ &= (1-3|x|)^{\beta(\gamma-1)} \\ &\geq 2((\lambda-1)c(1-3|x|)^{\sigma})^{\gamma-1} \\ &\geq 2w^{\gamma-1}.\end{aligned}$$

□

4.2.3 Proof of Lemma 4.2.1

That the Riesz potential of a radial function is radial is clear from symmetry considerations, and that it has the appropriate fractional Laplacian is classical: the Riesz potential serves the same role for the fractional Laplacian as the Newtonian potential does for the regular Laplacian (see, e.g., [21] or [16]).

The proof of this lemma uses facts from the theory of Riesz potentials. The key facts we will use are given by the following theorem of Adams [1] (pp 772):

Theorem 4.2.6. *If* $f \in L^p(\mathbb{R}^n)$*,* $1 \leq p < \infty$*, then*

1. $I_{\alpha_1} f \in BMO$ *if and only if* $M_{\alpha_1} f \in L^\infty(\mathbb{R}^n)$*.*
2. $I_{\alpha_1} f \in BMO$ *implies* $I_{\alpha_1+\alpha_2} f \in C^{\alpha_2}$ *where* $0 < \alpha_2 < 1$*.*

where

$$(M_{\alpha_1} f)(x) = \sup_{r>0} r^{\alpha_1 - n} \int_{B_r(x)} |f(z)| dz$$

is the fractional Hardy-Littlewood maximal function.

The plan is to set $\psi(x) = (1 - 3|x|)^{\beta-2\sigma}\chi_{B_{\frac{1}{3}}}(x)$, show that $\psi \in L^1(\mathbb{R}^n)$, and subsequently that $M_{\beta-2\sigma}\psi \in L^\infty$, whence we can apply the theorem to get the desired result.

First, we prove that $\psi \in L^1$:

$$\begin{aligned}\int \psi(x)dx &= C_n \int_0^{\frac{1}{3}} (1 - 3r)^{\beta-2\sigma} r^{n-1} dr \\ &\leq C_n \int_0^{\frac{1}{3}} (1 - 3r)^{\beta-2\sigma} dr \leq C\end{aligned}$$

since $\beta - 2\sigma > -\sigma > -1$.

Next, we consider the fractional maximal function. It is clear that the points of concern lie directly atop the singularity, that is, $r = \frac{1}{3}$. In balls B_ρ about such points, we see that

$$\int_{B\rho(\frac{1}{3})} \psi(x)dx \leq C\rho^{n+\beta-2\sigma},$$

which is precisely the scaling needed to see that $M_{\beta-2\sigma}\psi \leq C$.

Hence, we can apply the theorem of Adams and we conclude our lemma.

4.3 C^β estimates for u

The goal of this subsection is to provide a proof for Corollaries 4.0.12 and 4.0.13.

For the first, we will do this by analyzing the effective equation satisfied by u, restricted to Γ, in the neighborhood of a free boundary point. The estimates follow the spirit of the analysis conducted in Section III of [19]: we will first show that appropriate Holder norms of u satisfy certain pointwise estimates in terms of the value of u itself, and then put these estimates together to obtain a uniform C^β estimate.

For the second, we will follow a similar procedure, first using interior estimates to get pointwise bounds on ∇u when $y > 0$, and then tie these together with the C^β estimate along $\{y = 0\}$ to get a uniform C^β estimate.

4.3.1 Along $\{y = 0\}$

Lemma 4.3.1. $\{u > 0\} \cap \Gamma$ *is open with respect to* $\mathbb{R}^n$*, on which u satisfies*

$$\lim_{y\to 0} y^a \partial_y u = \gamma u^{\gamma-1},$$

and furthermore $u \in C^\infty(\{u > 0\})$*, such that the tangential derivatives of u, which we represent by* $\nabla_x u$*, satisfy*

$$|\nabla_x u(p,0)| \leq C(u(p,0))^{\frac{\beta-1}{\beta}},$$

and, moreover, that the tangential second derivatives of u, which we represent by $\nabla_{xx} u$*, satisfy*

$$|\nabla_{xx} u(p,0)| \leq C(u(p,0))^{\frac{\beta-2}{\beta}}$$

where p is any point in $\{u > 0\} \cap \{y = 0\}$.

Proof. If $p \in \Gamma$ is some point such that $u(p,0) > 0$, then the variant Boundary Harnack inequality tells us that $u(p,\delta) \geq Cu(p,0)$. If we make the usual dilation by

λ about p, we see that $u_\lambda(x,y) = \frac{1}{\lambda^\beta}u(\lambda(x-p)+p, \lambda y)$ satisfies

$$u_\lambda(p,1) > c\frac{1}{\lambda^\beta}u(p,0).$$

For λ sufficiently small, $c\lambda^{-\beta}u(p,0)$ can be made larger than the constant needed in Lemma 4.0.10, whence $u_\lambda \geq C > 0$ in $B_{\frac{1}{6}}(p,0)$, or, in the original u, we can say $u \geq C\lambda^\beta > 0$ in $B_{\frac{\lambda}{6}}(p,0)$ for λ sufficiently small.

Hence, the set $u > 0$ is open with respect to Γ, and, on every set D compactly contained within $\{u > 0\}$, is bounded away from zero, hence

$$\lim_{y\to 0} y^a \partial_y u = \gamma u^{\gamma-1} \in L^\infty(D).$$

Consider now $B = B_{\frac{1}{6}}(p,0)$. We let u_1 be the Riesz potential of $-\gamma u_\lambda^{\gamma-1}\chi_B$, and set $u_2 = u_\lambda - u_1$. Then on B we have

$$\lim_{y\to 0} y^a \partial_y u_1 = \gamma u_\lambda^{\gamma-1},$$

while

$$\lim_{y\to 0} y^a \partial_y u_2 = 0,$$

and both of the u_i satisfy $\nabla \cdot (y^a \nabla u_i) = 0$.

Since the tangential derivatives of u_2 also satisfy the same equations as u_2, we have that $u_2 \in C^\infty(B)$, with

$$|\nabla_x u_2(p,0)| \leq C,$$

and

$$|\nabla_{xx} u_2(p,0)| \leq C.$$

in $B_{\frac{1}{8}}(p,0)$, by the estimates found in [9] (see Proposition 3.4.5). Similarly, we can use the potential-theoretic estimates found in [20] iteratively to show that $u_1 \in C^\infty(B)$, with

$$|\nabla_x u_1(p,0)| \leq C,$$

and

$$|\nabla_{xx} u_1(p,0)| \leq C.$$

Hence, after rescaling, we can say that, for the tangential derivatives of u, we have

$$|\nabla_x u(p,0)| \leq C\lambda^{\beta-1},$$

and

$$|\nabla_{xx} u(p,0)| \leq C\lambda^{\beta-2}.$$

How small need λ be? Our condition was that $c\lambda^{-\beta}u(p,0) \geq c_0$, whence we see that $\lambda = (Cu(p,0))^{\frac{1}{\beta}}$ suffices. The conclusion follows. □

Up to now, it has been possible to treat the cases where $\beta \geq 1$ and $\beta < 1$ as if they were the same. For the remaining two theorems, we have to recognize the difference. The result here is proved very much the style of [19] and [6].

Theorem 4.3.2. *Suppose $\beta < 1$. Then there exists a $K = K(\delta, n, \beta)$, such that if $x_1, x_2 \in \mathbb{R}^n$ are in a δ-neighborhood of the free boundary, we have*

$$|u(x_1,0) - u(x_2,0)| \leq K|x_1 - x_2|^\beta.$$

If $\beta \geq 1$, there exists a $K = K(\delta, n, \beta)$, such that if $x_1, x_2 \in \mathbb{R}^n$ are in a δ-neighborhood of the free boundary, we have

$$|\nabla_x u(x_1, 0) - \nabla_x u(x_2, 0)| \leq K|x_1 - x_2|^{\beta-1}.$$

In either case, since away from the δ-neighborhood of the free boundary, $u \in C^\infty$, this means we can put the two together to get a uniform C^β norm for u.

Proof. As in the previous lemma, we notice that there is a constant C_1, such that if $u(x_1, 0) \geq C_1$, then the variant boundary Harnack inequality tells us that u satisfies the conditions for Lemma 4.0.10, and hence $u \geq C_2$ inside $B_{\frac{1}{6}}(x_1, 0)$. Rescaling this statement, we have that if $u(x_1, 0) \geq C_1 r^\beta$, then $u \geq C_2 r^\beta$ inside $B_{\frac{r}{6}}(x_1, 0)$.

We now consider three cases:

1. $u(x_1, 0) \geq C_1(6|x_1 - x_2|)^\beta$ and $|x_1 - x_2| < \frac{\delta}{4}$.
2. $u(x_1, 0) \geq C_1(6|x_1 - x_2|)^\beta$ and $|x_1 - x_2| \geq \frac{\delta}{4}$.
3. $u(x_1, 0), u(x_2, 0) \leq C_1(6|x_1 - x_2|)^\beta$.

1) Consider the line segment joining x_1 and x_2, with $r = 6|x_1 - x_2|$. Since $u(x_1, 0) \geq C_1 r^\beta$, we have $u(x, 0) \geq C_2 r^\beta$ inside $B_{\frac{r}{6}}(x_1, 0)$, which happily is precisely $B_{|x_1 - x_2|}(x_1, 0)$. Hence, when $\beta < 1$, the mean value theorem applied along this line segment tells us

$$|u(x_1, 0) - u(x_2, 0)| \leq |\nabla_x u(x', 0)||x_1 - x_2|,$$

where x' is some point along our line segment. By applying the estimates from Lemma 4.3.1, we have

$$|\nabla_x u(x', 0)||x_1 - x_2| \leq C(u(x', 0))^{\frac{\beta-1}{\beta}}|x_1 - x_2| \leq C|x_1 - x_2|^{\beta}.$$

When $\beta \geq 1$, we consider instead

$$|\nabla_x u(x_1, 0) - \nabla_x u(x_2, 0)| \leq |\nabla_{xx} u(x', 0)||x_1 - x_2|,$$

and by applying the estimates on the tangential second derivatives, we have

$$|\nabla_{xx} u(x', 0)||x_1 - x_2| \leq C(u(x', 0))^{\frac{\beta-2}{\beta}}|x_1 - x_2| \leq C|x_1 - x_2|^{\beta-1}.$$

2) In this case, we simply say directly that, if $\beta < 1$,

$$\begin{aligned} |u(x_1, 0) - u(x_2, 0)| &\leq |u(x_1, 0)| + |u(x_2, 0)| \\ &\leq C\delta^{\beta} \leq C|x_1 - x_2|^{\beta}, \end{aligned}$$

where we invoke Theorem 4.0.11 on the last step. If $\beta \geq 1$, we say

$$\begin{aligned} |\nabla_x u(x_1, 0) - \nabla_x u(x_2, 0)| &\leq |\nabla_x u(x_1, 0)| + |\nabla_x u(x_2, 0)| \\ &\leq C(u(x_1, 0))^{\frac{\beta-1}{\beta}} + C(u(x_2, 0))^{\frac{\beta-1}{\beta}} \\ &\leq C\delta^{\beta-1}, \end{aligned}$$

where we invoke Theorem 4.0.11 on the last step.

3) The calculations are exactly like case 2, only instead of invoking Theorem 4.0.11 to bound u pointwise, we invoke the hypothesis. $\square$

From this result, Corollary 4.0.12 is obvious.

4.3.2 The estimates when $y > 0$

Note first that inside $y > 0$, $u \in C^\infty$, since $\nabla \cdot (y^a \nabla u) = 0$ is uniformly elliptic with smooth coefficients on any compact subset contained within $\{y > 0\}$ (with differing ellipticities, of course). We can thus assume that u is smooth far away, and concentrate on its behavior for small values of y.

We start with an elementary lemma that gives us pointwise estimates on the derivatives of u via rescaling:

Lemma 4.3.3. *Let u be a non-negative function satisfying $\nabla \cdot (y^a \nabla u) = 0$ inside $B_R \cap \{y > 0\}$ for some large R . Then there is a constant C depending only on n and σ, such that*

$$|\nabla u(x_0, y_0)| \leq \frac{C}{y_0} u(x_0, y_0),$$

and

$$|D^2 u(x_0, y_0)| \leq \frac{C}{y_0^2} u(x_0, y_0)$$

whenever $y_0 > 0$ and $B_{\frac{y_0}{2}}((x_0, y_0)) \subset B_R$.

Proof. Suppose first that $y_0 = 1$ and $B_{\frac{1}{2}}(x_0, 1)$ is inside B_R. Then inside this ball, y^a is a bounded, C^∞ coefficient, so the standard regularity theory for weak solutions gives us the estimates

$$|\nabla u(x_0, 1)| \leq C u(x_0, 1),$$

and

$$|D^2 u(x_0, 1)| \leq C u(x_0, 1).$$

For general y, we simply consider the rescaling $w(x, y) = u(x_0 + (x - x_0)y_0, y_0 y)$ and write the estimate for w in terms of u. $\square$

Next, we provide a boundary estimate on the growth of u away from the line $y = 0$. We choose nice constants for the varius radii and the lines, bearing in mind that we can rescale.

Lemma 4.3.4. *Let u be an energy minimizer inside $B_8 \cap \{y > 0\}$ with nontrivial free boundary. Then there exists a constant C such that, for $y < 1$ and $(x, y) \in B_3$, we have*

$$|u(x, y) - u(x, 0)| \leq Cy^{\beta}.$$

Proof. If $u(x, 0) = 0$ for any $(x, 0) \in B_3$, then Theorem 4.0.11 suffices for that value of x. Thus we only need consider values of x such that $u(x, 0) > 0$. For these values of x, we have

$$\lim_{y \to 0} y^a \partial_y u(x, y) = \gamma u^{\gamma - 1},$$

which we choose to rewrite as

$$\lim_{y \to 0} y^a \partial_y (u^{2-\gamma}) = C_\gamma.$$

A bit of calculation shows us that $u^{2-\gamma}$ has $\nabla \cdot (y^a \nabla (u^{2-\gamma})) \geq 0$. Let $w = u^{2-\gamma}$ along $y = 0$ and $y = 1$ and also along $\partial_{B_8} \cap \{0 < y < 1\}$, and satisfy $\nabla \cdot (y^a \nabla w) = 0$ inside. Then the maximum principle tells us that $w \geq u^{2-\gamma}$ since $u^{2-\gamma}$. Since u has a nontrivial free boundary, Theorem 4.0.11 tells us that u, and hence w, is bounded along $y = 1$, say, $w|_{y=1} \leq C'$. Then that tells us that

$$w(x, y) \leq w(x, 0) + C' y^{2\sigma}$$

inside $B_3 \cap \{0 < y < 1\}$. Hence,

$$|u^{2-\gamma}(x,0) - u^{2-\gamma}(x,y)| \le C' y^{2\sigma}.$$

Applying an elementary inequality, we have the desired result. $\square$

With these two lemmata in hand, we can prove the analogue of Theorem 4.3.2 for the domain where $y > 0$.

Theorem 4.3.5. *Suppose $\beta < 1$. Then there exists a $K = K(\delta, n, \beta)$, such that if $X_1 = (x_1, y_1), X_2 = (x_2, y_2) \in \mathbb{R}^{n+1}_+$ are in a δ-neighborhood of the free boundary, we have*

$$|u(X_1) - u(X_2)| \le K|X_1 - X_2|^\beta.$$

If $\beta \ge 1$, there exists a $K = K(\delta, n, \beta, \alpha)$, such that if $X_1, X_2 \in \mathbb{R}^{n+1}_+$ are in a δ-neighborhood of the free boundary, we have

$$|u(X_1) - u(X_2)| \le K|X_1 - X_2|^\alpha$$

for any $\alpha < 1$.

Proof. Without loss of generality, assume that $y_1 \le y_2$.

First, assume that $\beta < 1$. Suppose that $y_2 \le |X_1 - X_2|^{1-\beta}$. Then, using the C^β-regularity of u restricted to $y = 0$ and the previous lemma, we write that

$$\begin{aligned} |u(x_1,y_1) - u(x_2,y_2)| &\le |u(x_1,y_1) - u(x_1,0)| + |u(x_2,y_2) - u(x_2,0)| + |u(x_1,0) - u(x_2,0)| \\ &\le C(y_1^\beta + y_2^\beta) + C|x_1 - x_2|^\beta \\ &\le 2C|X_1 - X_2|^{\frac{\beta}{1-\beta}} + C|X_1 - X_2|^\beta \\ &\le C|X_1 - X_2|^\beta. \end{aligned}$$

On the other hand, if $y_1 \geq |X_1 - X_2|^{1-\beta}$, then we use our pointwise gradient estimates and the special properties of this case to write that

$$\begin{aligned} |u(X_1) - u(X_2)| &\leq |\nabla u(\tilde{X})||X_1 - X_2| \\ &\leq \frac{u(\tilde{X})}{y_1}|X_1 - X_2| \\ &\leq \frac{C}{|X_1 - X_2|^{1-\beta}}|X_1 - X_2| \leq C|X_1 - X_2|^{\beta}, \end{aligned}$$

where $\tilde{X}$ is some point on the line joining X_1 and X_2.

If $y_1 \leq |X_1 - X_2|^{1-\beta}$ and $y_2 \geq |X_1 - X_2|^{1-\beta}$, then we consider:

$$|u(X_1) - u(X_2)| \leq |u(x_1, y_1) - u(x_2, |X_1 - X_2|^{1-\beta})| + |u(x_2, |X_1 - X_2|^{1-\beta}) - u(x_2, y_2)|.$$

The first term is controlled by the first method above, and the second term is controlled by the second method.

For the case when $\beta \geq 1$, simply let replace $|X_1 - X_2|^{1-\beta}$ in the preceding argument by $|X_1 - X_2|^{1-\alpha}$. $\square$

Chapter 5

Non-degeneracy

Our goal in this section is to prove that energy minimizers of

$$J(u) = \frac{1}{2}\int_{B_+} y^a |\nabla u|^2 dxdy + \int_\Gamma u^\gamma dx$$

possess the property they are non-degenerate, which is to say that near the free boundary, they grow away from 0, and do not stay small. To be precise, our final theorem is

Theorem 5.0.6. *Let 0 be a point of the free boundary of u, a minimizer of $J(u)$. Then there exists a constant $C > 0$ such*

$$\sup_{B_r} u \geq Cr^\beta.$$

Our strategy for proving this theorem is first to show that at a fixed distance away from the free boundary, there is a point which attains the desired growth.

Theorem 5.0.7. *Let $x_0 \in \Gamma$ be a point such that $d(x_0, F(u)) = r$, where $F(u)$ is the free boundary. Then, there exists a universal constant $\tau(n, \sigma, \gamma) > 0$ such that*

$$u(x_1) \geq \tau r^\beta,$$

where $|x_0 - x_1| \leq \frac{r}{4}$.

Proof. As is typical, we shall rely on the scaling property of energy minimizers, specifically, that on λB_+, $\frac{1}{\lambda^\beta} u(x_0 + \lambda X)$ is still an energy minimizer. Hence, we can assume $d(x_0, F(u)) = 1$ and we only need to show that there exists an x_1 with

$$u(x_1) \geq c_1,$$

for some x_1 with $|x_0 - x_1| \leq \frac{1}{4}$.

The standard Green's identity applied to some test function $\phi \geq 0$ with support compactly contained within $B_1(x_0)$ tells us that

$$\int_\Gamma u(\lim_{y\to 0} y^a \partial_y \phi) - \phi(\lim_{y\to 0} y^a \partial_y u) dx = -\int_{B_{\frac{1}{2}}(u_0)\cap\{y>0\}} u\nabla \cdot (y^a \nabla \phi) dxdy.$$

We notice that, along $\Gamma \cap B_{\frac{1}{2}}(x_0)$, we have $u > 0$, and hence

$$\lim_{y\to 0} y^a \partial_y u = \gamma u^{\gamma-1}.$$

Thus, we attain the condition that

$$\left|\int_\Gamma \gamma u^{\gamma-1} \phi dx\right| \leq \left|\int_\Gamma u \lim_{y\to 0} y^a \partial_y \phi dx\right| + \left|\int u(x) \nabla \cdot (y^a \nabla \phi) dxdy\right|.$$

Now let us suppose to the contrary that there is no constant c_1, that is to say, for any $\varepsilon > 0$ there is a minimizer such that $|u| \leq \varepsilon$ inside $B_{\frac{1}{2}}(x_0)$. Since $d(x_0, F(u)) = 1$, we have $d(B_{\frac{1}{2}}(x_0), F(u)) \leq \frac{3}{2}$, and we apply optimal regularity to bound the interior term: on $B_{\frac{1}{2}}(x_0) \cap \{y > 0\}$, we have

$$u(x) \leq C.$$

Putting these conditions together, and we get the argument that

$$\left|\int_\Gamma \gamma \varepsilon^{\gamma-1} \phi dx\right| \leq \left|\int_\Gamma \varepsilon \lim_{y\to 0} y^a \partial_y \phi dx\right| + \left|\int \varepsilon \nabla \cdot (y^a \nabla \phi) dxdy\right|$$

for arbitrarily small ε. Since the left hand side becomes very large and the right hand side goes to zero, we have a contradiction: u cannot be made uniformly arbitrarily small inside $B_{\frac{1}{2}}(x_0) \cap \Gamma$, and thus there exists a constant τ such that $u > \tau$ at some point, which we call x_1. $\square$

Now we begin the proof of Theorem 5.0.6, which is essentially identical to that given in [5], and reproduced here for completeness:

Proof. The proof is divided into two steps.

Step 1. Let u be a local minimizer in B_M such that

- 0 is a free boundary point,
- $B_1(e_1, 0) \cap \Gamma \subset \{u > 0\} \cap \Gamma$,
- $u(e_1, 0) = \tau > 0$ where τ is the constant from Theorem 5.0.7, known to be bounded both from above and from below away from 0.

We claim the existence of $\lambda > 0$ and $M > 0$ universal, the latter being large, such that

$$\sup_{B_M \cap \Gamma} u \geq (1 + \lambda)\tau.$$

Suppose not. This implies the existence of a sequence of energy minimizers for our problem, $(u_k)_{k\in\mathbb{N}}$, satisfying the three listed conditions, such that

$$\lim_{k\to\infty} \sup_{B_M \cap \Gamma} u = \tau.$$

From our regularity theorems, the family $(u_k)_k$ is equicontinuous, and may be assumed to converge uniformly on every compact subset of $\mathbb{R}^{n+1}_+$ to a function u_∞ which satisfies $\lim_{y\to 0} y^a \partial_y u_\infty \geq 0$. Moreover, $u_\infty(\cdot, 0)$ has a maximum at e_1, thus it is constant from the maximum principle. Hence $u_\infty \equiv \tau$, a contradiction because 0 is a free boundary point.

Step 2. Assume that 0 is a free boundary point. As in [8], we construct inductively a sequence of points $(x_m)_m \in \mathbb{R}^n$, such that

- $u(x_{m+1}, 0) \geq (1+\lambda)u(x_m, 0)$.
- If $r_m = d(x_m, \{u = 0\})$ and $\tilde{x_m}$ is a free boundary point realizing the distance, we have $x_{m+1} \in B_{Mr_m}(\tilde{x_m})$ with $u(x_{m+1}, 0) \geq \tau r_m^\beta$. This is from the construction of Step 1 applied to the rescaling $\frac{1}{r_m^\beta} u(\tilde{x_m} + r_m x, r_m y)$.

In particular, we have

$$|x_{m+1} - x_m| \leq 2(M+1)r_m.$$

We end the induction at the first point x_m which leaves B_1. This is possible, since the sequence $u(x_m, 0)$ grows geometrically in m, but is controlled by optimal regularity considerations. Let m_0 be the index of the first point to leave B_1. Then we write

$$\begin{aligned}
u(x_{m_0+1}, 0) &= \sum_{m=0}^{m_0} (u(x_{m+1}, 0) - u(x_m, 0)) \geq \lambda \sum u(x_m, 0) \\
&\geq C\lambda \sum d(x_m, \{u = 0\} \cap B_1)^\beta \text{ by Theorem 5.0.7} \\
&\geq C' \sum |x_{m+1} - x_m|^\beta \\
&\geq C''\lambda \sum |x_{m+1} - x_m| \text{ because } |x_{m+1} - x_m| \leq 1 \\
&\geq C'''.
\end{aligned}$$

The last step is justified because C'', λ are both universal, and m_0 is bounded universally by the geometric growth of the construction. Hence, for all $r > 0$, we have

$$\sup_{B_{Mr}} u \geq C''' r^{\beta},$$

which by rescaling Mr to r was precisely what we set out to prove. □

Corollary 5.0.8. *In terms of n-dimensional Lebesgue measure, the positivity set* $\{u > 0\}$ *has positive density, bounded away from 0, in a neighborhood of any free boundary point. That is to say,*

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} \geq \delta(n, \sigma, \gamma) > 0$$

for any ball B_r centered about a free boundary point.

Proof. This is a consequence of nondegeneracy, which says that a sufficiently positive point exists, and of the Holder continuity of u (Theorem 4.3.2). □

Bibliography

[1] David R. Adams. A note on Riesz potentials. *Duke Math. J.*, 42(4):765–778, 1975.

[2] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[3] H. W. Alt and D. Phillips. A free boundary problem for semilinear elliptic equations. *J. Reine Angew. Math.*, 368:63–107, 1986.

[4] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa. Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana Univ. Math. J.*, 30(4):621–640, 1981.

[5] L. Caffarelli, J. Roquejoffre, and Y. Sire. Variational problems with free boundaries for the fractional laplacian. *J. Eur. Math. Soc. (JEMS)*, 12(5):1151–1179, 2010.

[6] L. A. Caffarelli and D. Kinderlehrer. Potential methods in variational inequalities. *J. Analyse Math.*, 37:285–295, 1980.

[7] L.A. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[8] Luis Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[9] Luis A. Caffarelli, Sandro Salsa, and Luis Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[10] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[11] E. Fabes, D. Jerison, and C. Kenig. The Wiener test for degenerate elliptic equations. *Ann. Inst. Fourier (Grenoble)*, 32(3):vi, 151–182, 1982.

[12] E. B. Fabes, C. E. Kenig, and D. Jerison. Boundary behavior of solutions to degenerate elliptic equations. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 577–589. Wadsworth, Belmont, CA, 1983.

[13] Eugene B. Fabes, Carlos E. Kenig, and Raul P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[14] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1997.

[15] David Kinderlehrer and Guido Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.

[16] N. S. Landkof. *Foundations of modern potential theory*. Springer-Verlag, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[17] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:43–77, 1963.

[18] Daniel Phillips. Hausdorff measure estimates of a free boundary for a minimum problem. *Comm. Partial Differential Equations*, 8(13):1409–1454, 1983.

[19] Daniel Phillips. A minimization problem and the regularity of solutions in the presence of a free boundary. *Indiana Univ. Math. J.*, 32(1):1–17, 1983.

[20] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[21] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[22] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

Vita

Ray Yang was born in Los Angeles, California in August of 1981, to Paul Yang and Sun-Yung Alice Chang. He received his Bachelor of Arts degree in physics from Princeton University. He was accepted and started graduate studies in mathematics at the University of Texas in August of 2005.

Permanent address: RayAYang@gmail.com

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.